The Geometry[1]

First Book

Problems which can be constructed without employing anything but circles and straight lines.

All the problems of geometry can be easily reduced to such terms that there is no need beyond that of knowing the length of certain straight lines to construct them.[2]

How the calculation of arithmetic relates to the operations of geometry.

And as all arithmetic is composed of only four or five operations, which are addition, subtraction, multiplication, division, and the extraction of roots, which can be taken for a species of division,[3] so there is nothing else to do in geometry concerning the lines that are sought, to prepare them to be known, than adding or subtracting others [to or from] them; or else having one, which I shall call the unit to relate so much more to numbers, and which can ordinarily be taken at one's discretion, then having yet two others, to find a fourth, which would be to one of the two as the other is to the unit, which is the same as multiplication; or else to find a fourth, which would be to one of these two, as the unit is to the other, which is the same as division; or finally to find one or two or more mean proportionals between the unit and some other line, which is the same as to draw out the square root or cube root, etc. And I do not fear to introduce these terms from arithmetic into geometry, to render myself more intelligible.

Multiplication.

Let, for example, AB be the unit and let it be necessary to multiply BD by

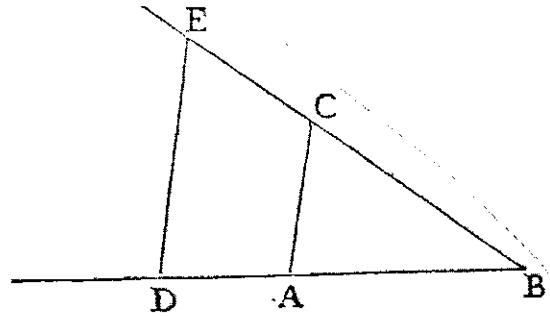
---

[1] *The Geometry* was originally published as one of three appendices to the *Discourse on Method*. The other two appendices (or "attempts with this method" [*essais de cette Méthode*], as Descartes described them; see also the top of p. 55 for this same expression) were the *Dioptrics* and the *Meteorology*.

[2] It may be noted at the start that *The Geometry* addresses itself to problems. The student may wish to consider what the difference between a problem and a theorem is. (Cf. Heath, *Euclid's Elements*, Vol. 1, pp. 124-129.) Descartes does not defend his claim in this treatise, and perhaps the reader will doubt whether it holds universally. For an example of a problem whose solution does not clearly require the knowledge of the length of some straight line, consider the inscription of a regular pentagon in a circle (*Elements* IV, 10 & 11).

[3] Is Descartes taking arithmetic to mean the totality of arithmetic problems here? What about arithmetic theorems? Or is the problem/theorem distinction not applicable to arithmetic? This last question suggests another, namely, How closely related are geometry and arithmetic? (Cf. *Rules for the Direction of the Mind*, Rule IV.)

BC, I have only to join the points A and C, then to draw DE parallel to CA, and BE is the product of this multiplication.[4]

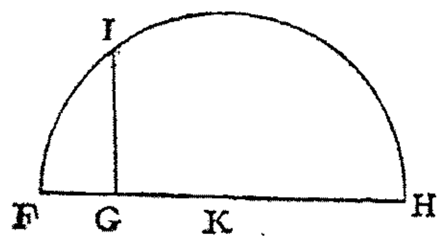


Division.

Or else if it is necessary to divide BE by BD, having joined the points E and D, I draw AC parallel to DE, and BC is the product of this division.

The extraction of the square root.

Or else if it is necessary to draw the square root of GH, I add FG, which is the unit, to it in a straight line, and dividing FH in two equal parts at the point K, from the center K, I draw the circle FIH, then setting up from the point G a straight line so far as I, at right angles to FH, GI is the root sought. I say nothing here of the cube root, nor of others, because I shall speak of them more conveniently later.



How one can use ciphers in geometry.[5]

But often one has no need to trace the lines thus on paper and it suffices to designate them by some letter, each by only one. So to add the line BD to GH, I name the one *a* and the other *b*, and I write $a + b$; and $a - b$, to subtract *b* from *a*; and *ab*, to multiply them, the one by the other; and *a/b*, to divide *a* by *b*; and *aa*, or a^2, to multiply *a* by itself; and a^3, to multiply it once more by *a*, and thus to infinity; and $\sqrt{a^2+b^2}$, to draw the square root of a^2+b^2; and $\sqrt[3]{a^3-b^3+abb}$, to draw the cube root of a^3-b^3+abb, and thus with others.

Here it is to be noted that by a^2 or b^3 or the like, I ordinarily conceive only lines altogether simple, although, to use the names employed in the algebra, I name them square or cubes, etc.

It is also to be noted that all the parts of one and the same line ought ordinarily to be expressed by as many dimensions, the one as the other, when the unit is not at all determined in the question, as here a^3 contains as many as *abb* or b^3 of which the line which I have named $\sqrt[3]{a^3-b^3+abb}$ is composed; but that it is not the same when the unit is determined, because it can be understood everywhere where there are too many or too few dimensions; so if

[4] On Descartes's method of multiplication, see Exercises, p. 84.

[5] The reader may find it illuminating at this point to consult the *Oxford English Dictionary* for the meaning(s) of "cipher" (here translating "chiffre"), and also "symbol." Indeed, this and the previous section are greatly illuminated by *Rules for the Direction of the Mind*, rule 16.

it is necessary to draw the cube root from *aabb* - *b*, it is necessary to think that the quantity *aabb* is divided once by the unit and that the other quantity *b* is multiplied twice by the same.[6]

Moreover, in order not to fail to remember the names of these lines, a separate register must always be made to the extent that one posits them or changes them, writing, for example,

AB = 1, that is to say, AB equals 1.
GH = *a*
BD = *b*, etc.

How it is necessary to arrive at Equations which serve to resolve problems.

Thus wishing to resolve some problem, one must first consider it as already done and give names to all the lines which seem necessary to construct it, to those which are unknown as well as to the others. Then without considering any difference between the known lines and the unknown, we must go over the difficulty, according to the order which shows most naturally of all in what manner they depend mutually on each other, until we have found means of expressing one same quantity in two ways: what is named an equation, for the terms of one of these two ways are equal to that of the others. And we must find as many such equations as the lines we have supposed, which are unknown. Or else if so many are not found and yet we have omitted nothing of what is desired in the question, this testifies that it is not entirely determined. And then we can take known lines at our discretion for unknown lines to which no equation corresponds. After this if there remain still more,[7] we must use in order each of the equations which so remain, either by considering each alone or by comparing it with the others, to explain each of the unknown lines, and so disentangle them that there remains but one, equal to another, which is known, or else whose square or the cube or the square of the

---

[6] Descartes here uses the term "dimension" not to distinguish the line from the plane, but to distinguish a line from a line, depending on the generation of those lines. Thus, if *a* and *b* are lines, *ab* is a line generated from *a* and *b*. It has two dimensions, even though it is a line. This use of "dimension" is analogous to the modern algebraic term "degree." One is reminded at this point of the traditional doctrine that there is no comparison between magnitudes different in kind. Thus, one cannot add a line to a surface, nor can one say by how much the one exceeds the other. Descartes, however, relaxes this claim in problems in which unity is determined. It may be noted that division, being the converse of multiplication, lowers the dimension of the quotient. Thus, *ab/c* is of the first dimension. Descartes does not say what taking roots does to dimensionality. The reader may, perhaps, be able to determine this for himself.

[7] Descartes is discussing the technique for reducing several equations in several unknowns to a single equation in one unknown. The second part of the paragraph describes the resulting equation by employing the convention that unknown lines are expressed by letters at the end of the alphabet, '*z*' for instance, while 'known' or 'given' lines are represented by letters at the beginning of the alphabet ('*a*' through '*d*' in the text). Note that all the examples he gives have terms of the same dimension in a single equation.

square, or the supersolid, or the square of the cube, etc. is equal to something produced by the addition or subtraction of two or more other quantities, of which one is known and the others are composed of some mean proportionals between the unit and this square, or cube, or square of the square, etc., multiplied by other knowns. I write this in this manner:

$z = b$, or

$z^2 = -az + bb$, or

$z^3 = +az^2 + bbz - c^3$, or

$z^4 = az^3 - c^3 z + d^4$, etc.

This is to say, z, which I take for the unknown quantity, is equal to b, or the square of z is equal to the square of b less a multiplied by z. Or the cube of z is equal to a multiplied by the square of z plus the square of b multiplied by z less the cube of c. And so with the others.

And we can always so reduce all the unknown quantities to one alone, when the problem can be constructed by circles and straight lines, or also by conic sections, or even by another line which is only one or two degrees more composed.[8] But I do not stop to explain this in more detail because I would take away from you the pleasure of learning it of yourself and the utility of cultivating your mind by exerting it, which, in my opinion, is the principal thing that we can draw from this science. Also because I note nothing so difficult that those who are a little versed in common geometry and in algebra, and who attend to all that is in this treatise, cannot find.

And so I content myself here with warning you that, provided that in disentangling these equations we do not fail to make use of all the divisions which are possible, we will infallibly have the most simple terms to which the question can be reduced.

What are the plane problems.

And if it can be resolved by ordinary geometry, that is to say, by using only straight and circular lines traced on a plane surface, when the last equation shall have been entirely disentangled, all that will remain at most is one unknown square, equal to something produced by the addition or subtraction of its root multiplied by some known quantity and of some other quantity also known.

How they are resolved.

And then this root, or unknown line, is easily found. For if I have, for

[8] The term 'degree' is consistently used by Descartes to modify curves, not terms. At present he has not defined the degree of a curve. It will turn out that conic sections are first degree curves. Descartes does not justify this claim here. Moreover, it is a bit puzzling, in that he has just above associated constructability in general with the possibility of reducing the several equations in several unknowns to a single equation in one unknown. Now he seems to be limiting the constructions to those carried out by certain curves.